Electromagnetic field objects in terms of Balance of Geometric flows.

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Abstract

This paper reviews our physical motivation for choosing appropriate formal presentation of electromagnetic field objects (EMFO). Our view is based on the understanding that EMFO are spatially finite entities carrying internal dynamical structure, so, their available integral time stability should be represented by appropriate adaptation of their internal dynamical structure to corresponding local stress-energy-momentum balance relations with other physical objects. This adaptation process has two aspects: internal and external. Clearly, finding adequate internal dynamical structure giving appropriate integral characteristics of the object, will bring also appropriate behavior of EMFO as a whole. Therefore, the internal local stress-energy-momentum balance among the subsystems of EMFO should formally be presented by appropriately defined tensor-field quantities, which are meant to suggest a dinamical understanding of the abilities of EMFO to successfully, or not successfully, communicate with all the rest physical world.

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1 Introduction

Modern theoretical view on classical fields accepts that time dependent and space propagating electromagnetic fields are flows of time stable physical entities which have been called in the early 20th century *photons*. Since appropriate in this respect nonlinearizations of Maxwell vacuum equations are still missing [1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15], and the seriously developed quantum theory also does not give appropriate, from our viewpoint, description of time stable entities of electromagnetic field nature, we decided to look back to the rudiments of the electromagnetic theory trying to reconsider its assumptions in order to come to equations giving appropriate solutions, in particular, solutions, demonstrating internal dynamical structure, having finite spatial carrier at every moment of their existence, and space-propagating as a whole, keeping, of course, their physical identity and recognizability.

According to our view, in presence of appropriate environment, the dynamical equations, describing locally, i.e., around every point inside the spatial carrier, the evolution of the object, may come from giving an explicit form of the quantities, controlling the local internal and external exchange processes, in other words, the equations must express corresponding *local balance relations*.

We note that the properties *spatial finiteness* and *internal dynamical structure* we consider as very essential ones. So, the classical material points and the infinite classical fields (e.g. plane waves) should not be considered as models of physical objects since the former have no structure and cannot be destroyed at all, and the latter carry infinite energy, so they cannot be finite-time

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created. Therefore, the Born-Infeld "principle of finiteness" [2] stating that *"a satisfactory theory should avoid letting physical quantities become infinite"* we strengthen as follows:

All real physical objects are spatially finite entities and NO infinite values of the physical quantities carried by them should be allowed.

Clearly, together with the purely qualitative features, physical objects carry important quantitatively described physical properties, and any external interaction may be considered as an exchange of the corresponding quantities provided both, the object and the corresponding environment, carry them. Hence, the more universal is a physical quantity the more useful for us it is, and this moment determines the exclusively important role of stress-energy-momentum, which modern physics considers as the most universal one, so we may assume that:

Propagating electromagnetic field objects necessarily carry energy-momentum.

Further in the paper we shall follow the rule:

Physical recognizability of time-stable subsystems of a physical system requires corresponding mathematical recognizability in the theory.

Assuming that any physical interaction presupposes dynamical flows of some physical quantities among the subsystems of the physical system considered, the field nature of the objects suggests *local nature* of these flows, so, **every continuous subsystem is supposed to be able to build CORRESPONDING LOCAL INSTRUMENTS, realizing explicitly the flows**. In static cases these flows reduce, of course, to *stress*. Formally this means:

1. We must have a mathematical field object $\mathcal{A}$ representing the system as a whole.

2. The supposed existence of recognizable and mutually interacting subsystems $(A_1, A_2, ...)$ of $\mathcal{A}$ leads to the assumption for *real* but *admissible*, i.e., not leading to annihilation, changes of the subsystems, so, such changes should be formally represented by tensor objects.

3. The local flow manifestation of the admissible real changes suggests to make use of appropriate combination of tensor objects, corresponding tensor co-objects, and appropriate invariant differential operators.

4. Every coupling inside this combination shall distinguish existing partnership, i.e., interaction, among the subsystems, so, all such couplings should be duly respected when the system's time-stability and spatial propagation are to be understood.

In order to make our view more easily and rightly apprehended we begin with the strongly idealized example of a *static* classical field object, if for *mathematical images* of the *physical constituents*, further called *formal constituents*, of the object are chosen vector fields on the traditional classical space $\mathbb{R}^3$.

2 Maxwell stress tensors

Every vector field, defined on an arbitrary manifold M, generates 1-parameter family φ_t of (local in general) diffeomorphisms of M. Therefore, having defined a vector field X on M, we can consider for each $t \in \mathbb{R}$ the corresponding diffeomorphic image $\varphi_t(U)$ of any region $U \subset M$. Hence, interpreting the external parameter t as time, which is NOT obligatory, vector fields may be formally tested as formal images of the dynamical constituents of some spatially finite real field objects.

Let now X be a vector field on the euclidean space $(\mathbb{R}^3, g)$, where g is the euclidean metric in $T\mathbb{R}^3$, having in the canonical global coordinates $(x^1, x^2, x^3 = x, y, z)$ components $g_{11} = g_{22} = g_{33} = 1$, and $g_{12} = g_{13} = g_{23} = 0$. The induced euclidean metric in $T^*\mathbb{R}^3$ has in the dual bases the same components and will be denoted further by the same letter g. The corresponding isomorphisms between the tangent and cotangent spaces and their tensor, exterior and symmetric products will be denoted by the same simbol $\tilde{g}$, so (summation on the repeating indecies is assumed)

$$\tilde{g}\left(\frac{\partial}{\partial x^i}\right) = g^{ik}\left(\frac{\partial}{\partial x^k}\right) = dx^i, \quad (\tilde{g})^{-1}(dx^i) = \frac{\partial}{\partial x^i} \quad \cdots$$

Having a co-vector field, i.e. 1-form α, or another vector field Y on $\mathbb{R}^3$, we can form the *flow of X across* α, or across the $\tilde{g}$-coobject $\tilde{g}(Y)$ of Y:

$$i_X\alpha = \langle \alpha, X\rangle = \alpha_1 X^1 + \alpha_2 X^2 + \alpha_3 X^3,$$

$$i_X\tilde{g}(Y) = \langle \tilde{g}(Y), X\rangle = g(X,Y) \equiv X.Y = g_{ij}X^iY^j = X_iY^i = X_1Y^1 + X_2Y^2 + X_3Y^3.$$

This flow of X is an invariant entity, so to its admissible and appropriate changes should be paid due respect. According to classical vector analysis on $\mathbb{R}^3$ [16] for the differential of the function $g(X,Y)$ we can write

$$(\tilde{g})^{-1}[\mathbf{d}g(X,Y)] = (X.\nabla)Y + (Y.\nabla)X + X \times curl(Y) + Y \times curl\,(X),$$

where in our coordinates

$$X.\nabla = \nabla_X = X^i\frac{\partial}{\partial x^i}, \quad (X.\nabla)Y = \nabla_X Y = X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j},$$

"$\times$" denotes the usual vector product, and

$$(curl X)^i = \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}, \frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}, \frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\right).$$

The Hodge $*_g$-operator acts in these coordinates as follows:

$$*dx = dy \wedge dz, \quad *dy = -dx \wedge dz, \quad *dz = dx \wedge dy,$$

$$*(dx \wedge dy) = dz, \quad *(dx \wedge dz) = -dy, \quad *(dy \wedge dz) = dx,$$

$$*(dx \wedge dy \wedge dz) = 1, \quad *1 = dx \wedge dy \wedge dz.$$

Corollary. The following relation holds ($\mathbf{d}$ denotes the exterior derivative):

$$curl(X) = (\tilde{g})^{-1} * \mathbf{d}\,\tilde{g}(X), \quad \text{or} \quad \tilde{g}(curl X) = *\,\mathbf{d}\,\tilde{g}(X).$$

Assume now that in the above expression for $\mathbf{d}g(X,Y)$ we put $X = Y$, i.e., we consider the invariant local change of the flow of X across its proper coobject $\tilde{g}(X)$. We obtain

$$\frac{1}{2}\mathbf{d}g(X,X) = \frac{1}{2}\mathbf{d}(X^2) = \tilde{g}(X \times curl X + (X.\nabla)X) = \tilde{g}(X \times curl X + \nabla_X X).$$

In components, the last term on the right reads

$$(\nabla_X X)^j = X^i\nabla_i X^j = \nabla_i(X^iX^j) - X^j\nabla_i X^i = \nabla_i(X^iX^j) - X^j \mathrm{div}\, X,$$

where (denoting by L_X the Lie derivative along X)

$$\mathrm{div} X = *L_X\,(dx\wedge dy\wedge dz) = *\left(\frac{\partial X^i}{\partial x^i}dx\wedge dy\wedge dz\right) = \frac{\partial X^i}{\partial x^i}.$$

Substituting into the preceding relation, replacing $\mathbf{d}(X^2)$ by $(\nabla_i\delta^i_j X^2)dx^j$, where δ^i_j is the unit tensor in $T\mathbb{R}^3$, and making some elementary transformations we obtain

$$\nabla_i\left(X^iX^j-\frac{1}{2}g^{ij}X^2\right) = \left[(curl\,X)\times X + X\mathrm{div}\,X\right]^j.$$

The symmetric 2-tensor

$$M^{ij} = X^iX^j - \frac{1}{2}\,g^{ij}X^2 = \frac{1}{2}\Big[X^iX^j + \big(\tilde{g}^{-1}\circ *\tilde{g}(X)\big)^{ik}\big(*\,\tilde{g}(X)\big)_k{}^j\Big] \tag{1}$$

we shall call further Maxwell stress tensor generated by the (arbitrary) vector field $X\in\mathfrak{X}(\mathbb{R}^3)$. The components M^i_j represent the generated by the dynamical nature of X local stresses, and the local stress energy is represented in terms of $tr(M^i_j)=M^i_i$. Momentum is missing since time is a missing dimension. *The appropriate changes of M^i_j, given in this idealised case by $(curl\,X\times X)$ and $(X\,div\,X)$, could be considered as possible local instruments in terms of which, in presense of more constituents and recognizable subsystems, corresponding balance relations to be written down.*

We specially note that, formally, M^{ij} may be represented as sum of the stresses carried by X and by the 2-vector $\tilde{g}^{-1}*\tilde{g}(X)=\tilde{g}^{-1}i_X(dx\wedge dy\wedge dz)$. Hence, since $\tilde{g}^{-1}*\tilde{g}(X)$ is uniquely determined by X and g, we may consider an idealised physical object, built of two constituents X and $\tilde{g}^{-1}*\tilde{g}(X)$. It should be noted however that *interacting stress* between the two costituents is missing: M^{ij} is sum of the stresses carried by X and $\tilde{g}^{-1}*\tilde{g}(X)$.

Clearly, when we raise and lower indices in canonical coordinates with $\tilde{g}$ we shall have the following component relations:

$$M_{ij}=M^j_i=M^{ij}$$

which does not mean, of course, that we equalize quantities being elements of different linear spaces.

We note now some formal relations.

First, the easily verified relation between the vector product "$\times$" and the wedge product in the space of 1-forms on $\mathbb{R}^3$:

$$X\times Y = (\tilde{g})^{-1}\left(*\left(\tilde{g}(X)\wedge\tilde{g}(Y)\right)\right)$$
$$=(\tilde{g})^{-1}\circ i(X\wedge Y)(dx\wedge dy\wedge dz),\quad X,Y\in\mathfrak{X}(\mathbb{R}^3).$$

We are going to consider now the differential flow nature of $\nabla_iM^i_jdx^j$.

Proposition. If $\alpha=\tilde{g}(X)$ then the following relation holds ($i(X)\mathbf{d}\alpha$ means $X^i\mathbf{d}\alpha_{ij}dx^j$):

$$\tilde{g}(curl\,X\times X) = i(X)\mathbf{d}\alpha = -*(\alpha\wedge *\mathbf{d}\alpha).$$

Proof.

$$\tilde{g}(curl\,X\times X) = \tilde{g}\circ(\tilde{g})^{-1}\ *\left(\tilde{g}(curl X)\wedge\tilde{g}(X)\right) = -*(\alpha\wedge *\mathbf{d}\alpha).$$

For the component of $i(X)\mathbf{d}\alpha$ before dx we obtain

$$-X^2\left(\frac{\partial\alpha_2}{\partial x^1}-\frac{\partial\alpha_1}{\partial x^2}\right) - X^3\left(\frac{\partial\alpha_3}{\partial x^1}-\frac{\partial\alpha_1}{\partial x^3}\right),$$

and the same quantity is easily obtained for the component of $[-*(\alpha\wedge *\mathbf{d}\alpha)]$ before dx. The same is true for the components of the two 1-forms before dy and dz. The proposition is proved. Since

a 2-form may be considered as a 2-volume form on $U^2 \subset \mathbb{R}^3$, we may interpret the above relation in the sense that, $\mathbf{d}\alpha = \mathbf{d}\tilde{g}(X)$ is a volume 2-form across which the vector field X will drag the points of the finite region $U^2 \subset \mathbb{R}^3$.

As for the second term $X\mathrm{div}X$ of the divergence $\nabla_i M^{ij}$, since $\mathbf{d} * \alpha = \mathrm{div}X(dx \wedge dy \wedge dz)$, we easily obtain

$$i(\tilde{g}^{-1}(*\alpha))\mathbf{d} * \alpha = (\mathrm{div}X)\alpha. \tag{2}$$

Hence, analogically, the 2-vector $\tilde{g}^{-1}(*\alpha)$ will drag the points of $U^2 \subset \mathbb{R}^3$ across the 3-form $\mathbf{d} * \alpha$, meaning that the 1-form α changes to $(\mathrm{div}X)\alpha$.

We can write now

$$M_{ij} = \frac{1}{2}[\alpha_i\alpha_j + (*\alpha)_i{}^k(*\alpha)_{kj}], \quad \nabla_i M^i_j = \left[i(X)\mathbf{d}\alpha + i(\tilde{g}^{-1}(*\alpha))\mathbf{d} * \alpha\right]_j. \tag{3}$$

So, a stress balance between the formal constituents X and $\tilde{g}^{-1}(*\alpha)$ is described by

$$i(X)\mathbf{d}\alpha = -i(\tilde{g}^{-1}(*\alpha))\mathbf{d} * \alpha. \tag{4}$$

A formal suggestion that comes from the above relations (4) is:

The interior product of a (multi)vector and a differential form (i.e. the flow of a (multi)vector field across a differential form) may be considered as appropriate means, generating quantitative measure of local physical interaction.

Hence, the naturally isolated two terms in $\nabla_i M^i_j dx^j$ suggest: any realizable static stress, that can be associated with the vector field X, to be described by $(\alpha, *\alpha)$, and the recognizable nature of α and $*\alpha$ to be guaranteed by the balance equation $i(X)\mathbf{d}\alpha = -i(\tilde{g}^{-1}(*\alpha))\mathbf{d} * \alpha$, or by

$$i(X)\mathbf{d}\alpha = 0, \quad i(\tilde{g}^{-1}(*\alpha))\mathbf{d} * \alpha = 0, \quad \text{i.e.,} \qquad X \times \mathit{curl}\, X = \mathit{0}, \ \mathrm{div}\, X = \mathit{0}.$$

Recalling now how the Lie derivative with respect to (multi)vectors acts on 1-forms and 2-forms [17], namely,

$$L_X\alpha = \mathbf{d}\langle\alpha, X\rangle + i(X)\mathbf{d}\alpha, \quad \text{i.e.,} \qquad L_X\alpha - \mathbf{d}\langle\alpha, X\rangle = i(X)\mathbf{d}\alpha,$$

$$L_{*\bar{\alpha}} * \alpha = \mathbf{d}\langle *\alpha, *\bar{\alpha}\rangle - (-1)^{deg(*\bar{\alpha}).deg(\mathbf{d})} i(*\bar{\alpha})\mathbf{d} * \alpha, \quad \text{i.e.} \quad L_{*\bar{\alpha}} * \alpha - \mathbf{d}\langle *\alpha, *\bar{\alpha}\rangle = -i(*\bar{\alpha})\mathbf{d} * \alpha,$$

where $deg(*\bar{\alpha}) = 2, \ deg(\mathbf{d}) = 1$, we see that the flow of X across the 2-form $\mathbf{d}\alpha$, and the flow of the 2-vector $\tilde{g}^{-1}(*\alpha)$ across $\mathbf{d} * \alpha$, are given by the difference between two well defined coordinate free quantities, and this difference determines when the local change $\mathbf{d}\alpha$ of α with respect to X, resp. $\mathbf{d} * \alpha$ of $*\alpha$ with respect to $\tilde{g}^{-1}(*\alpha)$, cannot be represented by $\mathbf{d}\langle\alpha, X\rangle$, resp. $\mathbf{d}\langle *\alpha, \tilde{g}^{-1}(*\alpha)\rangle$.

We pass now to the case of two formal constituents represented by *two* vector fields.

Let V and W be two vector fields on our euclidean 3-space. Summing up the corresponding two Maxwell stress tensors (Sec.1) we obtain the identity:

$$\nabla_i M^{ij}_{(V,W)} \equiv \nabla_i \left(V^iV^j + W^iW^j - g^{ij}\frac{V^2 + W^2}{2} \right) =$$
$$= \left[(\mathit{curl}\ V) \times V + V\mathrm{div}\ V + (\mathit{curl}\ W) \times W + W\mathrm{div}\ W\right]^j.$$

Note that the balance in this case **may** look like, for example, as follows:

$$(\mathit{curl}\ V) \times V = -W\mathrm{div}\ W, \quad (\mathit{curl}\ W) \times W = -V\mathrm{div}\ V,$$

which suggests internal/mutual stress balance between two subsystems created by two constituents formally described by V and W.

Let now $(a(x,y,z), b(x,y,z))$ be two arbitrary functions on $\mathbb{R}^3$. We consider the transformation

$$(V,W) \to (V\,a - W\,b, V\,b + W\,a).$$

We specially note:

1. The tensor $M_{(V,W)}$ transforms to $(a^2+b^2)M_{(V,W)}$.

2. The transformations $(V,W) \to (V\,a - W\,b, V\,b + W\,a)$ do not change the eigen directions structure of $M^{ij}_{(V,W)}$.

3. If $a = \cos\theta, b = \sin\theta$, where $\theta = \theta(x,y,z)$ then the tensor $M_{(V,W)}$ stays invariant:

$$M_{(V,W)} = M_{(V\cos\theta - W\sin\theta, V\sin\theta + W\cos\theta)}.$$

The expression inside the parenteses above, denoted by $M^{ij}_{(V,W)}$, looks formally the same as the introduced by Maxwell tensor $M^{ij}(\mathbf{E},\mathbf{B})$ from physical considerations concerned with the electromagnetic stress energy properties of continuous media in presence of external electromagnetic field $(\mathbf{E},\mathbf{B})$. Formally any vector V, or any couple of vectors (V,W), define such tensor (which we denoted by M_V, or $M_{(V,W)}$), called further **Maxwell stress tensor**. The term, "stress" in this general mathematical setting could be justified by the above mentioned dynamical nature of vector fields. It deserves noting here that the two-vector case should be expected to satisfy some conditions of compatability between V and W in order to physically represent some mutually balanced time stable stress flows.

We emphasize the following moments:

1. The differential identity satisfied by $M_{(V,W)}$ is purely mathematical;

2. On the two sides of this identity stay well defined coordinate free quantities;

3. The tensors $M_{(V,W)}$ do NOT introduce interaction stress: **the full stress is the sum of the stresses generated by each one of the constituents** (V,W).

Physically, we may say that the corresponding physical medium that occupies the spatial region $\mathbf{U}_o$ and is parametrized by the points of the mathematical subregion $U_o \subset \mathbb{R}^3$, is subject to *compatible* and *admissible* physical "stresses", and these physical stresses are quantitatively described by the corresponding physical interpretation of the tensor $M_{(V,W)}$. Clearly, we could extend the couple (V,W) to more vectors $(V_1, V_2, ..., V_p)$, but then the mentioned invariance properties of $M_{(V,W)}$ may be lost, or should be appropriately extended.

Finally, note that the stress tensor M^{ij} appears as been subject to the divergence operator, and if we interpret the components of M^{ij} as physical stresses, then its divergence acquires, in general, the physical interpretation of *force density.* Of course, in the *static* situation as it is given by the relation considered, no stress propagation is possible, so at every point the local forces mutually compensate: $\nabla_i M^{ij} = 0$.

Now, analizing the eigen and other properties of the Maxwell energy tensors, we try to find some appropriate suggestions.

3 What the properties of Maxwell stress tensors suggest.

We consider $M^{ij}(\mathbf{E},\mathbf{B})$ at some point $p \in \mathbb{R}^3$ and assume that in general the vector fields $\mathbf{E}$ and $\mathbf{B}$ are lineary independent, so $\mathbf{E}\times\mathbf{B} \neq 0$. Let the coordinate system be chosen such that the coordinate plane (x,y) to coincide with the plane defined by $\mathbf{E}(p), \mathbf{B}(p)$. In this coordinate system

$\mathbf{E} = (E_1, E_2, 0)$ and $\mathbf{B} = (B_1, B_2, 0)$, so, identifying the contravariant and covariant indices through the Euclidean metric g^{ij} (so that $M^{ij} = M^i_j = M_{ij}$), we obtain the following nonzero components of the stress tensor:

$$M^1_1 = (E^1)^2 + (B^1)^2 - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2); \quad M^1_2 = M^2_1 = E^1\, E_2 + B^1\, B_2;$$

$$M^2_2 = (E^2)^2 + (B^2)^2 - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2); \quad M^3_3 = -\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2).$$

Since $M^1_1 = -M^2_2$, the trace of M is $Tr(M) = -\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$.

The eigen value equation acquires the simple form

$$\big[(M^1_1)^2 - (\lambda)^2\big] + (M^1_2)^2\big](M^3_3 - \lambda) = 0.$$

The corresponding eigen values are

$$\lambda_1 = -\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2); \quad \lambda_{2,3} = \pm\sqrt{(M^1_1)^2 + (M^1_2)^2} = \pm\frac{1}{2}\sqrt{(I_1)^2 + (I_2)^2},$$

where $I_1 = \mathbf{B}^2 - \mathbf{E}^2$, $I_2 = 2\mathbf{E}.\mathbf{B}$.

The corresponding to λ_1 eigen vector Z_1 must satisfy the equation

$$\mathbf{E}(\mathbf{E}.Z_1) + \mathbf{B}(\mathbf{B}.Z_1) = 0,$$

and since the non-zero $(\mathbf{E}, \mathbf{B})$ are lineary independent, the two coefficients $(\mathbf{E}.Z_1)$ and $(\mathbf{B}.Z_1)$ must be equal to zero, therefore, $Z_1 \neq 0$ must be orthogonal to $\mathbf{E}$ and $\mathbf{B}$, i.e. Z_1 must be colinear to $\mathbf{E} \times \mathbf{B}$:

The other two eigen vectors $Z_{2,3}$ satisfy correspondingly the equations

$$\mathbf{E}(\mathbf{E}.Z_{2,3}) + \mathbf{B}(\mathbf{B}.Z_{2,3}) = \Big[\pm\frac{1}{2}\sqrt{(I_1)^2 + (I_2)^2} + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\Big] Z_{2,3}. \qquad (*)$$

Taking into account the easily verified relation

$$\frac{1}{4}\Big[(I_1)^2 + (I_2)^2\Big] = \left(\frac{\mathbf{E}^2 + \mathbf{B}^2}{2}\right)^2 - |\mathbf{E} \times \mathbf{B}|^2,$$

so that

$$\frac{\mathbf{E}^2 + \mathbf{B}^2}{2} - |\mathbf{E} \times \mathbf{B}| \geq 0\ ,$$

we conclude that the coefficient before $Z_{2,3}$ on the right is always different from zero, therefore, the eigen vectors $Z_{2,3}(p)$ lie in the plane defined by $(\mathbf{E}(p), \mathbf{B}(p)),\ p \in \mathbb{R}^3$. In particular, the above mentioned transformation properties of the Maxwell stress tensor $M(V, W) \rightarrow (a^2 + b^2)M(V, W)$ show that the corresponding eigen directions do not change under the transformation $(V, W) \rightarrow (V\,a - W\,b, V\,b + W\,a)$.

The above consideration suggests: *the intrinsically allowed dynamical abilities of the field object might be: translational along* $(\mathbf{E} \times \mathbf{B})$, *and rotational inside the plane defined by* $(\mathbf{E}, \mathbf{B})$, *hence, we may expect finding field objects the propagation of which shows intrinsic local compatability between rotation and translation.*

It is natural to ask now **under what conditions the very E and B may be eigen vectors of** $M(\mathbf{E}, \mathbf{B})$? Assuming $\lambda_2 = \frac{1}{2}\sqrt{(I_1)^2 + (I_2)^2}$ and $Z_2 = \mathbf{E}$ in the above relation and having in

view that $\mathbf{E}\times\mathbf{B}\neq 0$ we obtain that $\mathbf{E}(\mathbf{E}^2)+\mathbf{B}(\mathbf{E}.\mathbf{B})$ must be proportional to $\mathbf{E}$, so, $\mathbf{E}.\mathbf{B}=0$, i.e. $I_2=0$. Moreover, substituting now $I_2=0$ in that same relation we obtain

$$\mathbf{E}^2=\frac{1}{2}(\mathbf{B}^2-\mathbf{E}^2)+\frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2)=\mathbf{B}^2, \quad \text{i.e.,} \quad I_1=0.$$

The case "-" sign before the square root, i.e. $\lambda_3=-\frac{1}{2}\sqrt{(I_1)^2+(I_2)^2}$, leads to analogical conclusions just the role of $\mathbf{E}$ and $\mathbf{B}$ is exchanged.

Corollary 1. $\mathbf{E}$ and $\mathbf{B}$ may be eigen vectors of $M(\mathbf{E},\mathbf{B})$ only if $I_1=I_2=0$.

These considerations suggest that if $I_1=0$, i.e. $|\mathbf{E}|^2=|\mathbf{B}|^2$, and propagation takes place, then the energy density can be presented in terms of each of the two formal constituents, moreover, in this respect, both constituents have the same rights. Therefore, a local mutual energy exchange between any supposed two subsystems, formally represented by appropriate combinations of $\mathbf{E},\mathbf{B}$, is not forbidden in general, but, if it takes place, it must be *simultaneous* and in *equal quantities*. Hence, if $I_1=0$ and $I_2=2\mathbf{E}.\mathbf{B}=0$, internal energy redistribution between the two supposed subsystems of the field object would be allowed, but such an exchange should occur *without available interaction energy*.

The following question is also of interest: is it *physically* allowed to interprit each of the two vector fields $\mathbf{E},\mathbf{B}$ *not as formal constituents*, but as formal images of recognizable time-stable physical *subsystems* of an electromagnetic field object?

Trying to answer this question we note that the relation $\mathbf{E}^2+\mathbf{B}^2=2|\mathbf{E}\times\mathbf{B}|$ and the required time-recognizability during propagation (with velocity "c") of each subsystem of the field object suggest/imply also that *each of the two subsystems must be able to carry locally momentum and to exchange locally momentum with the other one*, since this relation means that the energy density is always strongly proportional to the momentum density magnitude $\frac{1}{c}|\mathbf{E}\times\mathbf{B}|$. Hence, the couple $(\mathbf{E},\mathbf{B})$ is able to carry momentum, but *neither* of the formal constituents $\mathbf{E},\mathbf{B}$ *is able to carry momentum separately*. Moreover, the important observation here is that, verious combinations constructed out of the formal constituents $\mathbf{E}$ and $\mathbf{B}$, e.g., $(\mathbf{E}\,cos\theta-\mathbf{B}\,sin\theta,\mathbf{E}\,sin\theta+\mathbf{B}\,cos\theta)$, where $\theta(x,y,z;t)$ is a functon, may be considered as possible representatives of the two recognizable subsystems since they carry the same energy $\frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2)$ and momentum $\frac{1}{c}|\mathbf{E}\times\mathbf{B}|$ densities.

We note also the following.

Let $(\mathbf{E},\mathbf{B})$ be nonzero and *lineary independent*, then the triple $(\mathbf{E},\mathbf{B},\mathbf{E}\times\mathbf{B})$ defines a frame and a g-coframe at every point, where the field object is different from zero. We denote the corresponding frame by $\mathcal{R}'$, so we can write $\mathcal{R}'=(\mathbf{E},-\varepsilon\mathbf{B},-\varepsilon\mathbf{E}\times\mathbf{B})$, where $\varepsilon=\pm1$.

Since the physical dimension of the third vector $\mathbf{E}\times\mathbf{B}$ is different from that of the first two, we introduce the factor γ according to:

$$\gamma=\frac{1}{\sqrt{\frac{\mathbf{E}^2+\mathbf{B}^2}{2}}}.$$

Making use of γ, we introduce the so called *electromagnetic frame* :

$$\mathcal{R}=\left[\gamma\mathbf{E},-\varepsilon\gamma\mathbf{B},-\varepsilon\gamma^2\mathbf{E}\times\mathbf{B}\right].$$

Hence, at every point we've got two frames: $\mathcal{R}$, and the dimensionles coordinate frame $\mathcal{R}_o=l_o\Big[\partial_x,\partial_y,\partial_z\Big]$, $dim\,l_o=length$, as well as the corresponding co-frames $\mathcal{R}^*$ and $\mathcal{R}^*_o=l_o^{-1}(dx,dy,dz)$. The corresponding transformation matrix $\mathcal{M}:\mathcal{R}_o\rightarrow\mathcal{R}$ is defined by the relation $\mathcal{R}_o.\mathcal{M}=\mathcal{R}$. So, we obtain

$$\mathcal{M}=\begin{vmatrix}\alpha E^1 & -\varepsilon\gamma B^1 & -\varepsilon\gamma^2(\mathbf{E}\times\mathbf{B})^1\\ \gamma E^2 & -\varepsilon\gamma B^2 & -\varepsilon\gamma^2(\mathbf{E}\times\mathbf{B})^2\\ \gamma E^3 & -\varepsilon\gamma B^3 & -\varepsilon\gamma^2(\mathbf{E}\times\mathbf{B})^3\end{vmatrix}.$$

Let's now see when the matrix $\mathcal{M}$ does not change the 3-volume $\omega = dx \wedge dy \wedge dz$. Such a property requires $det(\mathcal{M}) = 1$. Now, from linear algebra it is known that such matrices have determinants equal to $\gamma^4(\mathbf{E}\times\mathbf{B}).(\mathbf{E}\times\mathbf{B}) = [\gamma^2|\mathbf{E}||\mathbf{B}||sin(\mathbf{E},\mathbf{B})|]^2$. So, this requirenment reduces to $\gamma^2|\mathbf{E}\times\mathbf{B}| = 1$. This last equation is equivalent to

$$|\mathbf{E}|^2 - 2|\mathbf{E}||\mathbf{B}||sin\beta| + |\mathbf{B}|^2 = 0,$$

where β is the angle between $\mathbf{E}$ and $\mathbf{B}$. Expressing $|\mathbf{E}|$ as a function of $|\mathbf{B}|$ through solving this quadratic equation with respect to $|\mathbf{E}|$, we obtain

$$0 < |\mathbf{E}|_{1,2} = |\mathbf{B}||sin\beta| \pm |\mathbf{B}|\sqrt{sin^2\beta - 1}.$$

This inequality is possible only if $|sin^2\beta| = 1$, so, $|\mathbf{E}| = |\mathbf{B}|$, $\mathbf{E}.\mathbf{B} = 0$, and $\sqrt{det(\mathcal{M})} = \gamma^2|\mathbf{E}||\mathbf{B}|$.

Corollary 2. The *unimodular* nature of $\mathcal{M}$: $\det\mathcal{M} = 1$, requires $\mathbf{E}.\mathbf{B} = 0, \mathbf{B}^2 = \mathbf{E}^2$, so, an orthonormal nature of the frame $\mathcal{R}$.

In our view these important properties have to be kept in mind when searching adequate equations, satisfied by the mathematical images of the physical constituents of time dependent and space propagating electromagnetic objects.

According to the obove considerations we may assume the following view on *real* electromagnetic field objects:

Every real electromagnetic field object is built of two recognizable and dynamically compatible subsystems (A_1, A_2), the mathematical images of which can be algebraically represented in terms of $(\mathbf{E}, \mathbf{B})$, both these subsystems carry always the same quantity of stress-energy-momentum, guaranteeing in this way that, any mutual energy-momentum exchange between A_1 and A_2 will always be in equal quantities and simultanious.

4 Real electromagnetic field objects viewed as built of two vector field constituents, being organized in two recognizable and permanently interacting subsystems.

4.1 Some formal relations

We begin with some notations and easily verified relations. Let g denote the euclidean metric on $\mathbb{R}^3$. We introduce the following notations:

$$\tilde{g}(\mathbf{E}) = \eta, \quad \tilde{g}(\mathbf{B}) = \beta, \quad \tilde{g}^{-1}(*\eta) = \bar{*\eta}, \quad \tilde{g}^{-1}(*\beta) = \bar{*\beta}.$$

Then we obtain

$$\tilde{g}(curl\,\mathbf{E}\times\mathbf{E}) = i(\mathbf{E})\mathbf{d}\eta, \quad \tilde{g}(curl\,\mathbf{B}\times\mathbf{B}) = i(\mathbf{B})\mathbf{d}\beta,$$

$$\tilde{g}(curl\,\mathbf{E}\times\mathbf{B}) = i(\mathbf{B})\mathbf{d}\eta, \quad \tilde{g}(curl\,\mathbf{B}\times\mathbf{E}) = i(\mathbf{E})\mathbf{d}\beta,$$

$$\mathbf{E}\,\mathrm{div}(\mathbf{B}) = i(\bar{*\eta})\mathbf{d}*\beta, \quad \mathbf{B}\,\mathrm{div}(\mathbf{E}) = i(\bar{*\beta})\mathbf{d}*\eta.$$

Further we shall use the notations and relations from Sec.2.

We introduce now some new relations.

Let E and E^* be two dual real finite dimensional vector spaces. The duality between E and E^* allows to distinguish the following (anti)derivation. Let $h \in E$, then we obtain the derivation

$i(h)$, or i_h, in $\Lambda(E^*)$ of degree (-1) (sometimes called substitution/contraction/isertion operator, interior product, algebraic flow) according to:

$$i(h)(x^{*1} \wedge \cdots \wedge x^{*p}) = \sum_{i=1}^{p} (-1)^{(i-1)} \langle x^{*i}, h \rangle x^{*1} \wedge \cdots \wedge \hat{x^{*i}} \wedge \cdots \wedge x^{*p}.$$

Clearly, if $u^* \in \Lambda^p(E^*)$ and $v^* \in \Lambda(E^*)$ then

$$i(h)(u^* \wedge v^*) = (i(h)u^*) \wedge v^* + (-1)^p u^* \wedge i(h)v^*.$$

Also, we get

$$i(h)u^*(x_1, \ldots, x_{p-1}) = u^*(h, x_1, \ldots, x_{p-1}),$$
$$i(x) \circ i(y) = -i(y) \circ i(x).$$

This antiderivation is extended to a mapping $i(h_1 \wedge \cdots \wedge h_p) : \Lambda^m(E^*) \to \Lambda^{(m-p)}(E^*)$, $m \geqq p$, according to

$$i(h_1 \wedge h_2 \wedge \cdots \wedge h_p)u^* = i(h_p) \circ \cdots \circ i(h_1)\, u^*.$$

Note that this extended mapping is not an antiderivation, except for $p = 1$.

This mapping is extended to multivectors and exterior forms which are linear combinations: if $\Psi = \Psi_1 + \Psi_2 + ...$ is an arbitrary multivector on E and $\Phi = \Phi^1 + \Phi^2 + ...$ is an arbitrary exterior form on E^* then $i_\Psi \Phi$ is defined as extention by linearity, e.g.,

$$i(\Psi_1 + \Psi_2)(\Phi^1 + \Phi^2) = i(\Psi_1)\Phi^1 + i(\Psi_1)\Phi^2 + i(\Psi_2)\Phi^1 + i(\Psi_2)\Phi^2.$$

This extension of the interior product allows to extend the Lie derivative of a differential form α along a vector field X to a derivative of α along a multivector field T [17], according to

$$\mathcal{L}_T(\Phi) = \mathbf{d} \circ i_T \Phi - (-1)^{deg(T)} i_T \circ \mathbf{d}\Phi.$$

If $\mathcal{L}_T(\Phi) = 0$ this extension allows to consider T as a symmetry of α.

We construct now the φ-extended insertion operator. Let E_1 and E_2 be two real vector spaces with corresponding bases $e_i, i = 1, 2, ..., dimE_1$ and $k_j, j = 1, 2, ..., dimE_2$, $T = \mathfrak{t}^i \otimes e_i$ be a E_1-valued q-vector, $\Phi = \alpha^j \otimes k_j$ be a E_2-valued p-form with $q \leq p$, and $\varphi : E_1 \times E_2 \to F$ be a bilinear map into the vector space F. Now we define $i^\varphi_T \Phi \in \Lambda^{p-q}(M, F)$:

$$i^\varphi_T \Phi = i_{\mathfrak{t}^i} \alpha^j \otimes \varphi(e_i, k_j), \quad i = 1, 2, ..., dim(E_1),\ j = 1, 2, ..., dim(E_2). \tag{5}$$

Also, if T_1, T_2 are two multivectors and α, β are two forms then $(i \otimes i)_{T_1 \otimes T_2}(\alpha \otimes \beta)$ is defined by

$$(i \otimes i)_{T_1 \otimes T_2}(\alpha \otimes \beta) = i_{T_1}\alpha \otimes i_{T_2}\beta.$$

We can define now the φ-extended Lie derivative. Let M be a n-dimensional manifold, Φ be a E_1-valued differential $p-$form on M, T be a E_2-valued q-multivectir field on M, with $q \leq p$ and $\varphi : E_1 \times E_2 \to F$ be a bilinear map. The φ-extended Lie derivative

$$\mathcal{L}^\varphi_T : \Lambda^p(M, E_1) \times \mathfrak{X}^q(M, E_2) \to \Lambda^{p-q+1}(M, F)$$

is defined as follows [21]:

$$\mathcal{L}^\varphi_T(\Phi) = \mathbf{d} \circ i^\varphi_T \Phi - (-1)^{deg(T).deg(\mathbf{d})} i^\varphi_T \circ \mathbf{d}\Phi, \tag{6}$$

where $\mathbf{d}$ is the exterior derivative on M, so, $deg(\mathbf{d}) = 1$. This definition suggests to consider the tensor field T as a *local* φ-symmetry of the differential form Φ when $\mathcal{L}^\varphi_T(\Phi) = 0$.

4.2 Static case

We begin with the *strongly idealized* static case where the constituent is modelled by a vector field on $\mathbb{R}^3$, denoted by $\mathbf{E}$. In order to recognize this vector field among the other ones we introduce 1-dimensional vector space V_o, its dual V_o^*, with corresponding dual bases e and ε, so our formal representation of the constituent looks as $\mathbf{E}\otimes e$. In searching for a partner our field $\mathbf{E}$ defines its g-dual 1-form $\eta=\tilde{g}(\mathbf{E})$, and making use of the Hodge star $*$ defined by g, it finds its partner constituent in the 2-form $*\eta$, which is equal to $i_{\mathbf{E}}\omega$, $\omega=dx\wedge dy\wedge dz$. According to the above view the balance in this idealized case between $\mathbf{E}\otimes e$ and $*\eta\otimes\varepsilon$ should be given by

$$\mathcal{L}^{\varphi}_{(\mathbf{E}\otimes e)}(\eta\otimes\varepsilon)=\mathcal{L}^{\varphi}_{(\bar{*\eta}\otimes e)}(*\eta\otimes\varepsilon),$$

where φ in this case is just the coupling $\langle\varepsilon,e\rangle=1$ between e and ε. Expanding this relation we get

$$\mathbf{d}\langle\eta,\mathbf{E}\rangle+i_{\mathbf{E}}\mathbf{d}\eta=\mathbf{d}\langle*\eta,\bar{*\eta}\rangle-i_{\bar{*\eta}}\mathbf{d}*\eta.$$

Since $\langle\eta,\mathbf{E}\rangle=\langle*\eta,\bar{*\eta}\rangle$ we get

$$i_{\mathbf{E}}\mathbf{d}\eta=-i_{\bar{*\eta}}\mathbf{d}*\eta,\qquad i.e.,\qquad i_{\mathbf{E}}\mathbf{d}\eta+i_{\bar{*\eta}}\mathbf{d}*\eta=0,$$

which is, according to relation (3), just the zero value of the divergence of the determined by $\mathbf{E}$ Maxwell stress tensor: The flow of $\mathbf{E}$ across $\mathbf{d}\eta$ is balanced by the flow of $\bar{*\eta}$ across $\mathbf{d}*\eta$.

This very elementary example suggests that, even *only one* vector field, $\mathbf{E}$ in our case, in order to *survive as a static stress generating factor* in the 3-space, looks for a balancing parthner, it finds such one in $\bar{i_{\mathbf{E}}\omega}=\bar{*\eta}$, its $\tilde{g}$-images, and the *static* stress flow $i_{\bar{*\eta}}\mathbf{d}*\eta$.

We pass now to the case of *two* static stress generating vector fields, denoted by $\mathbf{E}$ and $\mathbf{B}$. The new moment now is that the resulted generated local stress, although *static*, may depend on the **mutual** influence between the generated two local stresses by each one of the fields.

According to the above notations the two vector fields $\mathbf{E}$ and $\mathbf{B}$ appear together with their co-vectors η and β. Now the 1-dimensional vector space V_o should be, naturally, replaced by a 2-dimensional vector space V and its dual V^*, euclidean metric h, and corresponding h-dual bases $\{e_1,e_2\}$ and $\{\varepsilon_1,\varepsilon_2\}$: $\langle\varepsilon^i,e_j\rangle=\delta^i_j,\ i,j=1,2$. So, $(\mathbf{E},\mathbf{B})$ define a subsystem $\bar{\Omega}$ by

$$\bar{\Omega}=\mathbf{E}\otimes e_1+\mathbf{B}\otimes e_2,\quad\text{ and its } g\text{-dual co-image}\quad \Omega=\eta\otimes e_1+\beta\otimes e_2.$$

Since now the volume form in V^* is essential and is given by $\varepsilon^1\wedge\varepsilon^2$, we are going to introduce the balancing partner field Σ in two steps. First, the flow of $\bar{\Omega}$ across $\omega\otimes\varepsilon^1\wedge\varepsilon^2$:

$$\Sigma'=i(\bar{\Omega})(\omega\otimes\varepsilon^1\wedge\varepsilon^2).$$

Explicitly

$$\Sigma'=i(\mathbf{E}\otimes e_1+\mathbf{B}\otimes e_2)(\omega\otimes\varepsilon^1\wedge\varepsilon^2)=i(\mathbf{E})\omega\otimes i(e^1)(\varepsilon^1\wedge\varepsilon^2)+i(\mathbf{B})\omega\otimes i(e^2)(\varepsilon^1\wedge\varepsilon^2)$$

$$=i(\mathbf{E})\omega\otimes\varepsilon^2-i(\mathbf{B})\omega\otimes\varepsilon^1=-*\beta\otimes\varepsilon^1+*\eta\otimes\varepsilon^2.$$

Now Σ is defined by passing to V-valued 2-form by

$$\Sigma=(id\otimes\tilde{h}^{-1})(\Sigma')=-*\beta\otimes e_1+*\eta\otimes e_2.$$

Finally, the balancing partner is represented by Σ and

$$\bar{\Sigma}=-\bar{*\beta}\otimes e_1+\bar{*\eta}\otimes e_2.$$

Now the corresponding local *static* stress balance relation must pay due respect to the way the two stress generating formal constituents $\mathbf{E}$ and $\mathbf{B}$ generate interaction: the interaction must take care of their identities through recognizing them as eigen vectors of the stress-energy tensor, so, they should carry the *same local stress*, therefore, their *static* exchange stress, i.e., stress balance, must be *simultaneous* and *in equal quantities* (Corollary 1). In view of this, paying due respect to these properties of mutual symmetry and compatibility, we choose φ to be the *symmetrized tensor product* denoted by " $\vee$ ", and write:

$$\mathcal{L}^{\vee}_{\bar{\Omega}}(\Omega)=\mathcal{L}^{\vee}_{\bar{\Sigma}}(\Sigma). \tag{7}$$

We obtain:

$$\mathcal{L}^{\vee}_{\bar{\Omega}}(\Omega)=[\mathbf{d}\langle\eta,\mathbf{E}\rangle+i(\mathbf{E})\mathbf{d}\eta]\otimes e_1\vee e_1+[\mathbf{d}\langle\beta,\mathbf{B}\rangle+i(\mathbf{B})\mathbf{d}\beta]\otimes e_2\vee e_2$$
$$+[\mathbf{d}\langle\eta,\mathbf{B}\rangle+i(\mathbf{B})\mathbf{d}\eta+\mathbf{d}\langle\beta,\mathbf{E}\rangle+i(\mathbf{E})\mathbf{d}\beta]\otimes e_1\vee e_2.$$

$$\mathcal{L}^{\vee}_{\bar{\Sigma}}(\Sigma)=[\mathbf{d}\langle*\beta,*\bar{\beta}\rangle-i(*\bar{\beta})\mathbf{d}*\beta]\otimes e_1\vee e_1+[\mathbf{d}\langle*\beta,*\bar{\beta}\rangle-i(*\bar{\eta})\mathbf{d}*\eta]\otimes e_2\vee e_2$$
$$+[-\mathbf{d}\langle*\eta,*\bar{\beta}\rangle+i(*\bar{\beta})\mathbf{d}*\eta-\mathbf{d}\langle*\beta,*\bar{\eta}\rangle+i(*\bar{\eta})\mathbf{d}*\beta]\otimes e_1\vee e_2.$$

So, the balance relation (7) gives the following three equations:

$$\mathbf{d}\langle\eta,\mathbf{E}\rangle+i(\mathbf{E})\mathbf{d}\eta=\mathbf{d}\langle*\beta,*\bar{\beta}\rangle-i(*\bar{\beta})\mathbf{d}*\beta,$$
$$\mathbf{d}\langle\beta,\mathbf{B}\rangle+i(\mathbf{B})\mathbf{d}\beta=\mathbf{d}\langle*\eta,*\bar{\eta}\rangle-i(*\bar{\eta})\mathbf{d}*\eta,$$
$$\mathbf{d}\langle\beta,\mathbf{E}\rangle+i(\mathbf{E})\mathbf{d}\beta+\mathbf{d}\langle\eta,\mathbf{B}\rangle+i(\mathbf{B})\mathbf{d}\eta=-\mathbf{d}\langle*\eta,*\bar{\beta}\rangle-\mathbf{d}\langle*\beta,*\bar{\eta}\rangle+i(*\bar{\beta})\mathbf{d}*\eta+i(*\bar{\eta})\mathbf{d}*\beta.$$

In view of the relations

$$\langle\eta,\mathbf{E}\rangle=g(\mathbf{E},\mathbf{E})=\mathbf{E}^2,\ \langle\beta,\mathbf{B}\rangle=g(\mathbf{B},\mathbf{B})=\mathbf{B}^2,\ \langle*\beta,*\bar{\beta}\rangle=\mathbf{B}^2,\ \langle*\eta,*\bar{\eta}\rangle=\mathbf{E}^2,$$
$$\langle\eta,\mathbf{B}\rangle=g(\mathbf{E},\mathbf{B})=\mathbf{E}.\mathbf{B},\ \langle\beta,\mathbf{E}\rangle=g(\mathbf{B},\mathbf{E})=\mathbf{B}.\mathbf{E},\ \langle*\beta,*\bar{\eta}\rangle=\mathbf{B}.\mathbf{E},\ \langle*\eta,*\bar{\beta}\rangle=\mathbf{E}.\mathbf{B},$$

the equations read

$$i(\mathbf{E})\mathbf{d}\eta+i(*\bar{\beta})\mathbf{d}*\beta=\mathbf{d}\langle\mathbf{B}^2-\mathbf{E}^2\rangle$$
$$i(\mathbf{B})\mathbf{d}\beta+i(*\bar{\eta})\mathbf{d}*\eta=\mathbf{d}\langle\mathbf{E}^2-\mathbf{B}^2\rangle,$$
$$i(\mathbf{E})\mathbf{d}\beta+i(\mathbf{B})\mathbf{d}\eta-i(*\bar{\eta})\mathbf{d}*\beta-i(*\bar{\beta})\mathbf{d}*\eta=-\mathbf{d}(\mathbf{4}\,\mathbf{E}.\mathbf{B})$$

From the first two equations it follows the equation, i.e., *the static balance equation*,

$$i(\mathbf{E})\mathbf{d}\eta+i(*\bar{\beta})\mathbf{d}*\beta+i(\mathbf{B})\mathbf{d}\beta+i(*\bar{\eta})\mathbf{d}*\eta=0,$$

which is the Maxwell local conservation law $\nabla_i M^i_j(\mathbf{E},\mathbf{B})=0$ for the stress tensor

$$M^i_j(\mathbf{E},\mathbf{B})=\mathbf{E}^i\mathbf{E}_j+\mathbf{B}^i\mathbf{B}_j-\frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2)\delta^i_j$$
$$=\frac{1}{2}\Big[\mathbf{E}^i\mathbf{E}_j+\big(\tilde{g}^{-1}\circ*\tilde{g}(\mathbf{E})\big)^{ik}\big(*\tilde{g}(\mathbf{E})\big)_{kj}+\mathbf{B}^i\mathbf{B}_j+\big(\tilde{g}^{-1}\circ*\tilde{g}(\mathbf{B})\big)^{ik}\big(*\tilde{g}(\mathbf{B})\big)_{kj}\Big].$$

Recalling (Sec.3) that if we require the vector fields $\mathbf{E}$ and $\mathbf{B}$ to define at every point *eigen directions* of $M(\mathbf{E},\mathbf{B})$ and *unimodular* nature of the generated electromagnetic matrix $\mathcal{M}$, $\mathcal{M}^*\omega=\omega$, where $\omega=dx\wedge dy\wedge dz$, then we should assume

$$\mathbf{E}^2=\mathbf{B}^2 \quad \text{and} \quad \mathbf{E}.\mathbf{B}=0.$$

It is now elementary to see that under these last assumptions our *static* balance relation $\mathcal{L}_{\bar{\Omega}}^{\vee}(\Omega)=\mathcal{L}_{\bar{\Sigma}}^{\vee}(\Sigma)$ reduces to

$$i_{\bar{\Omega}}^{\vee}\mathbf{d}\Omega=-i_{\bar{\Sigma}}^{\vee}\mathbf{d}\Sigma\,. \tag{8}$$

This suggests to consider $\bar{\Omega},\bar{\Sigma}$, or Ω,Σ, as formal images of two *subsystems* of the object considered, which subsystems demonstrate stable stress equilibrium: *any stress lost by the first one is fully accepted by the second one and vice versa*. This corresponds to the fact that there is NO interaction stress in $M_j^i(\mathbf{E},\mathbf{B})$: the whole stress is sum of the stresses carryied by Ω and Σ. The hidden "dynamical" aspect of this static equilibrium is clearly seen from the reduced three equations.

$$i(\mathbf{E})\mathbf{d}\eta+i(\overline{*\beta})\mathbf{d}*\beta=0,$$

$$i(\mathbf{B})\mathbf{d}\beta+i(\overline{*\eta})\mathbf{d}*\eta=0,$$

$$i(\mathbf{E})\mathbf{d}\beta+i(\mathbf{B})\mathbf{d}\eta-i(\overline{*\eta})\mathbf{d}*\beta-i(\overline{*\beta})\mathbf{d}*\eta=0.$$

4.3 Time dependent case

First we note that introducing *time* is considered here as a quantitative comparing the courses of two physically independent processes, the one of which we call *referent*, e.g., the progress of appropriate watch, then the other one attains significance of *parametrised* process.

Hence, we have to specially note that the *time* parameter t used in this subsection we consider as *external* to the spatial coordinates (x,y,z) parameter, and the corresponding referent process must NOT influence the parametrised process . Some formal consequences of this consideration should be noted:

- time-derivatives are NOT derivatives along $\mathbb{R}^3$-spatial vector fields and, by assumption, corresponding local commutation relation between $\frac{\partial}{\partial t}$ and $\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$ *always* holds,

- time-derivatives do *not* change the tensor nature of the differentiated object.

Naturally, from physical viewpoint, any observed time change of the above discussed stress balance in the *static* case should presume corresponding *influence*, leading to its violation, and, of course, of its formal representation - relation (8). Physically, it may be expected the electromagnetic field object described, to sirvive through some kind of *time "pulsating"* at the space points, or through a *propagation as a whole* in the 3-space, or, both. So: *the local static balance should be replaced by an appropriate intrincally compatible local dynamical and time dependent balance.* Hence, in order to survive, our object must be able to generate *appropriate* spatial changes inside any occupied spatial area. In particular, in order the eigen nature of the static stress tensor to be approprately kept as eigen nature of the new "propagational" stress-energy-momentum tensor, the two zero divergences $*\mathbf{d}*\eta=0, *\mathbf{d}*\beta=0$, might be not necessarily kept to hold, as the explicit form of static equations (8) at the end of the previous subsection allow.

To this time-dependence of the behaviour of our electromagnetic field oject we are going to give formal decription by means of finding appropriate change of the static equation (8).

Equation (8) formally postulates equivalence between two vector valued 1-forms, so, any introduced influence object, representing how the new time-dependent balance would look like, is

expected, formally, also to be 1-form, containing appropriately first order t-derivative(s) and valued in the same vector space. This allows a natural return to the static balance equation through setting this new oject equal to zero.

Also, since the available spatial differential operators in (8) are just of *first order*, it seems natural the corresponding formal influence object to contain time derivatives of **not higher** than first order. Clearly, in view of the flow nature of the objects across their own spatial change objects in the static relation (8), the influence object is expected to express formally also a flow, but a flow across *time differentiated* object. Moreover, it should be expected also this time dependence to generate direct mutual influence between the two now time-dependent subsystems. Finally, since time derivation must not change the tensor nature of the differentiated object, and since Ω is 1-form, then the 2-form Σ is the natural candidate to be t-differentiated. So, we may write

$$i_{\overset{\vee}{\Omega}}\mathbf{d}\Omega + i_{\overset{\vee}{\Sigma}}\mathbf{d}\Sigma = \frac{1}{c}\, i_{\overset{\vee}{\Omega}}\frac{\partial}{\partial t}\Sigma. \tag{9}$$

Denoting $ct = \xi$, this equation (9) gives the following three equations

$$i(\mathbf{E})\mathbf{d}\eta + i(*\bar{\beta})\mathbf{d}*\beta = -i(\mathbf{E})\left(\frac{\partial}{\partial\xi}*\beta\right),$$

$$i(\mathbf{B})\mathbf{d}\beta + i(*\bar{\eta})\mathbf{d}*\eta = i(\mathbf{B})\left(\frac{\partial}{\partial\xi}*\eta\right),$$

$$i(\mathbf{E})\mathbf{d}\beta + i(\mathbf{B})\mathbf{d}\eta - i(*\bar{\eta})\mathbf{d}*\beta - i(*\bar{\beta})\mathbf{d}*\eta = i(\mathbf{E})\left(\frac{\partial}{\partial\xi}*\eta\right) - i(\mathbf{B})\left(\frac{\partial}{\partial\xi}*\beta\right).$$

Having in view the expressions for the extended Lie derivatives we can rewrite these equatios as follows:

$$L_{\mathbf{E}}\eta - \mathbf{d}\langle\eta,\mathbf{E}\rangle - \left[L_{*\bar{\beta}}*\beta - \mathbf{d}\langle*\beta,*\bar{\beta}\rangle\right] = -i(\mathbf{E})\left(*\frac{\partial\beta}{\partial\xi}\right)$$

$$L_{\mathbf{B}}\beta - \mathbf{d}\langle\beta,\mathbf{B}\rangle - \left[L_{*\bar{\eta}}*\eta - \mathbf{d}\langle*\eta,*\bar{\eta}\rangle\right] = i(\mathbf{B})\left(*\frac{\partial\eta}{\partial\xi}\right)$$

$$L_{\mathbf{E}}\beta - \mathbf{d}\langle\beta,\mathbf{E}\rangle + L_{\mathbf{B}}\eta - \mathbf{d}\langle\eta,\mathbf{B}\rangle + L_{*\bar{\eta}}*\beta - \mathbf{d}\langle*\beta,*\bar{\eta}\rangle + L_{*\bar{\beta}}*\eta - \mathbf{d}\langle*\eta,*\bar{\beta}\rangle$$
$$= i(\mathbf{E})\left(*\frac{\partial\eta}{\partial\xi}\right) - i(\mathbf{B})\left(*\frac{\partial\beta}{\partial\xi}\right).$$

4.4 Space-time representation

In the frame of the space-time view on physical processes the introduced variable $\xi = ct$ is no more indepentent on the choice of physical frames with respect to which we introduce spatial coordinates and write down time-dependent formal relations. Now ξ is considered as appropriate coordinate, it generates local coordinate base vetor $\frac{\partial}{\partial\xi}$ and corresponding co-vector (or 1-form) $d\xi, \langle d\xi, \frac{\partial}{\partial\xi}\rangle = 1$. So, the 3-volume $\omega = dx\wedge dy\wedge dz$ naturally becomes a 3-form on the 4-dimensional spase-time $\mathbb{R}^4$, and is extended to the 4-volume $\omega_o = dx\wedge dy\wedge dz\wedge d\xi$. Our purpose now is to find appropriate 4-dimensional form of our balance law given by equation (9).

Recall our two basic objects: the vector valued differential 1-form $\Omega = \eta\otimes e_1 + \beta\otimes e_2$ and the vector valued differential 2-form $\Sigma = -*\beta\otimes e_1 + *\eta\otimes e_2$ being defined entirely in terms of objects previously introduced on $\mathbb{R}^3$. We want now these objects to depend on ξ as they depend on the spatial coordinates, so to be appropriately extended to objects on $\mathbb{R}^4$.

Note that the 2-form Σ is defined making use of the 1-form Ω and the 3-form $\omega = dx\wedge dy\wedge dz$. Now, the 4th dimension ξ generates the coordinate 1-form $d\xi$, so, Ω turns to $d\xi$ for help to extend to a 2-form on $\mathbb{R}^4$, which is done in the simplest way: $\Omega \rightarrow \Omega\wedge d\xi$. We are in position now to consider the difference $\Omega\wedge d\xi - \Sigma$.

$$\Omega\wedge d\xi - \Sigma = (\eta\wedge d\xi)\otimes e_1 + (\beta\wedge d\xi)\otimes e_2 + *\beta\otimes e_1 - *\eta\otimes e_2$$
$$= (*\beta + \eta\wedge d\xi)\otimes e_1 - (*\eta - \beta\wedge d\xi)\otimes e_2.$$

In this way we get two differential 2-forms on $\mathbb{R}^4$ naturally recognized by the basis vectors of the external vector spase V:

$$F = *\beta + \eta\wedge d\xi, \quad \text{and} \quad -G = *\eta - \beta\wedge d\xi,$$

moreover, these two 2-forms are clearly identified as vector components of *one* V-valued 2-form:

$$\mathbf{\Omega} = F\otimes e_1 + G\otimes e_2.$$

In order to define corresponding flow, as we did it in previous subsections, we have to construct $\bar{\mathbf{\Omega}}$. The corresponding 2-vectors $\bar{F}$ and $\bar{G}$ are easily introduced making use of the isomorphism between 2-forms and 2-vectors defined by the volume form $\omega_o = dx\wedge dy\wedge dz\wedge d\xi$ according to

$$G = -i(\bar{F})\omega_o, \quad F = i(\bar{G})\omega_o: \;\rightarrow \quad \bar{\mathbf{\Omega}} = \bar{F}\otimes e_1 + \bar{G}\otimes e_2.$$

Another approach is to try to find appropriate *linear* map $\psi : \Lambda^2(\mathbb{R}^4)\rightarrow\Lambda^2(\mathbb{R}^4)$ sending F to G. So, we write down the presumed linear equation $\psi(F) = G$:

$$\psi(i_{\mathbf{B}}\omega + \eta\wedge d\xi) = i_{\mathbf{E}}\omega - \beta\wedge d\xi.$$

The linear nature of this presumed equation allows to reduce now ψ to the basis vectors of $\Lambda^2(\mathbb{R}^4)$: $(dx\wedge dy, dx\wedge dz, dy\wedge dz, dx\wedge d\xi, dy\wedge d\xi, dz\wedge d\xi)$, which gives:

$$\begin{array}{ll}
\psi(dx\wedge dy) = -dz\wedge d\xi & \psi(dx\wedge d\xi) = dy\wedge dz \\
\psi(dx\wedge dz) = dy\wedge d\xi & \psi(dy\wedge d\xi) = -dx\wedge dz \\
\psi(dy\wedge dz) = -dx\wedge d\xi & \psi(dz\wedge d\xi) = dx\wedge dy.
\end{array}$$

Obviously, ψ must satisfy the condition $\psi\circ\psi = -id_{\Lambda^2(\mathbb{R}^4)}$. Clearly, such linear map should define *complex structure* in the space $\Lambda^2(\mathbb{R}^4)$. As is well known, the Hodge star operator $*$ in Minkowski space-time, is defined by the relation $\alpha\wedge *\beta = (-1)^{ind(\mathbf{g})}\mathbf{g}(\alpha,\beta)\omega_o$, where $\omega_o = dx\wedge dy\wedge dz\wedge d\xi$, α,β are forms of the same rank, $ind(\mathbf{g})$ specifies the number of minuses in canonical coordinates of the pseudometric used. In our case the Minkowski pseudometric $\mathbf{g}$ has in canonical coordinates the components: $\mathbf{g}_{\mu\mu} = (-1,-1,-1,1)$; $\mathbf{g}_{\mu\nu} = 0, \mu\neq\nu = 1,2,3,4$. It should be noted here, that an interior product $i_X\omega_o$ is not always equal to $*_{\mathbf{g}}\mathbf{g}(X)$, where X is a multivector. In our case of Minkowski space-time with this pseudo-metric it is easy to verify that for 2-forms and $\mathbf{g}$-corresponding 2-vectors we obtain:

$$*\,F = -i(\bar{F})\omega_o, \quad F = i(\overline{*F})\omega_o, \quad \text{i.e.,} \quad \bar{G} = \overline{*F}. \tag{10}$$

In view of this, further we may use any of these two expressions.

We turn now to the corresponding balance law. In view of the preliminary assumed relations $\mathbf{E}^2 = \mathbf{B}^2, \mathbf{E}.\mathbf{B} = 0$, i.e., $F\wedge *F = F\wedge F = 0$, it reeds

$$i^{\vee}_{\bar{\mathbf{\Omega}}}\mathbf{d}\mathbf{\Omega} = 0, \quad \bar{\mathbf{\Omega}} = (\tilde{\mathbf{g}})^{-1}(\mathbf{\Omega}), \tag{11}$$

i.e., the $\vee$-flow of $\bar{\mathbf{\Omega}}$ across its change $\mathbf{d\Omega}$ does NOT lead to losses. It has to be noted, however, that this balance law may be written down without making use of (pseudo)metric, the volume form ω_o serves sufficiently well.

Since now $G = *F = -i(\tilde{\mathbf{g}}^{-1}F)\omega_o = -i(\bar{F})\omega_o$, equation (11) gives the following three equations

$$i_{\bar{F}}\mathbf{d}F = 0, \quad i_{\overline{*F}}\mathbf{d}*F = 0, \quad i_{\bar{F}}\mathbf{d}*F + i_{\overline{*F}}\mathbf{d}F = 0. \tag{12}$$

If $\delta = *\mathbf{d}*$ is the corresponding coderivative operator on Minkowski spacetime, the first two equations of (12) are correspondingly equivalent to

$$(*F)_{\mu\nu}(\delta * F)^{\nu} = 0, \quad F_{\mu\nu}(\delta F)^{\nu} = 0,$$

so, all nonlinear solutions: $\delta F \neq 0, \delta * F \neq 0$ of these equations must satisfy $det||F_{\mu\nu}|| = 0$, i.e., $\mathbf{E}.\mathbf{B} = 0$, and together with the third equation

$$i_{\bar{F}}\mathbf{d}*F + i_{\overline{*F}}\mathbf{d}F = i(\overline{\delta * F})F + i(\overline{\delta F})(*F) = 0$$

this requirement for nonlinearity extends to $det||F \pm *F|| = 0$, which is equivalent to $\mathbf{E}^2 = \mathbf{B}^2$.

The first two equations appeared first in [18], and the third jooined later in [19,20].

It is easy now to varify that writing down (9) and (12) totally in terms of $(\mathbf{E},\mathbf{B})$ we shall obtain the equations given at the end of the previous subsection.

The corresponding stress-energy-momentum tensor of any solution of (12)

$$T_{\mu}{}^{\nu} = -\frac{1}{2}\left[F_{\mu\sigma}F^{\nu\sigma} + (*F)_{\mu\sigma}(*F)^{\nu\sigma}\right]$$

clearly notifies absence of interaction stress-energy between the two subsystems formally represented by F and $*F$: the whole stress-energy is the sum of these quantities carried by F and $*F$. Of course, this admits local exchange $F \leftrightarrow *F$ of these quantities of special kind: *simultanious* and *in equal quantities*.

Here is a special class of nonlinear solutions of (12):

$$F = \varepsilon u\, dx \wedge dz + u\, dx \wedge d\xi + \varepsilon p\, dy \wedge dz + p\, dy \wedge d\xi$$
$$*F = -p\, dx \wedge dz - \varepsilon p\, dx \wedge d\xi + u\, dy \wedge dz + \varepsilon u\, dy \wedge d\xi,$$

where

$$u = \Phi(x, y, \xi + \varepsilon z)\cos\left(-\varepsilon\kappa\frac{z}{\mathcal{L}_o} + const\right),$$

$$p = \Phi(x, y, \xi + \varepsilon z)\sin\left(-\varepsilon\kappa\frac{z}{\mathcal{L}_o} + const\right), \quad \mathcal{L}_o = const, \quad \varepsilon = \pm 1, \quad \kappa = \pm 1,$$

and Φ is *arbitrary* function of it's four arguments. The energy density of these solutions is given by Φ^2, so, spatially finite solutions are *allowed*. These solutions propagate along the coordinate z inside some spatially infinite *helical cylinder* and "rotate" left or right depending on the sign of the constant κ. Their length size along the direction of propagation is $2\pi\mathcal{L}_o$.

Remarks and Comments

Recalling relation (6) and the zero values of

$$F \wedge F = i_{\bar{F}}(*F)\omega_o = 2\mathbf{E}.\mathbf{B}\omega_o = 0, \quad \text{and} \quad F \wedge *F = -i_{\bar{F}}F\omega_o = (\mathbf{E}^2 - \mathbf{B}^2)\omega_o = 0,$$

we see that the first two equations of (12) clearly suggest to consider $\bar{F}$ es a local symmetry of the 2-form F, but NOT as a local symmetry of $*F$, as well as to consider $\bar{*F}$ as a local symmetry of the 2-form $*F$ but NOT as a local symmetry of F:

$$\mathcal{L}_{\bar{F}}F = \mathbf{d}(i_{\bar{F}}F) - (-1)^2 i_{\bar{F}}\mathbf{d}F = -i_{\bar{F}}\mathbf{d}F = 0,$$

$$\mathcal{L}_{\bar{*F}} * F = \mathbf{d}(i_{\bar{*F}} * F) - (-1)^2 i_{\bar{*F}}\mathbf{d} * F = -i_{\bar{*F}}\mathbf{d} * F = 0.$$

Also, since $\mathbf{d}\omega_o = 0$, we obtain

$$\mathcal{L}_{\bar{F}}\omega_o = \mathbf{d}(i_{\bar{F}}\omega_o) - (-1)^2 i_{\bar{F}}\mathbf{d}\omega_o = -\mathbf{d} * F, \quad \mathcal{L}_{\bar{*F}}\omega_o = \mathbf{d}(i_{\bar{*F}}\omega_o) - (-1)^2 i_{\bar{*F}}\mathbf{d}\omega_o = \mathbf{d}F.$$

The last relations give some other view on the relativistic form of Maxwell free field equations $\mathbf{d}F = 0, \mathbf{d} * F = 0$: the two *null* bivectors $\bar{F}$ and $\bar{*F}$ are local symmetries of the standard volume form ω_o on Minkowski spacetime, in this sense, these two relativistic Maxwell equations appear as extensions of the nonrelativistic equations $div\mathbf{E} = 0, div\mathbf{B} = 0$, the invariant sense of which is

$$L_{\bar{\mathbf{E}}}(dx \wedge dy \wedge dz) = 0, \quad L_{\bar{\mathbf{B}}}(dx \wedge dy \wedge dz) = 0,$$

i.e., the two fields $(\mathbf{E}, \mathbf{B})$ do not change locally the 3-volume. Moreover, if we follow modern *guage* formulation of relativistic charge-free Maxwell equations, then $\mathbf{d}F = 0$ is in advance assumed, and in view of relations (10) the only aditional equation should read $\mathcal{L}_{\bar{F}}\omega_o = \mathbf{d}(i_{\bar{F}}\omega_o) = -\mathbf{d} * F = 0$.

Turning back to the static *null field* case where Maxwell free field equations require $curl\,\mathbf{E} = curl\,\mathbf{B} = 0$, and recalling relations in subsec.4.1, we see that the first two nonlinear static equations at the end of subsec.4.2 allow $curl\,\mathbf{E} \neq 0,\ curl\,\mathbf{B} \neq 0$, i.e., $\mathbf{d}\eta \neq 0,\ \mathbf{d}\beta \neq 0$, since the 3d-matrices $\mathbf{d}\eta$ and $\mathbf{d}\beta$ are antisymmetric, and their determinants are *necessarily equal to zero*, which allows the components of $\mathbf{E}$ and $\mathbf{B}$ to be algebraically determined as nonzero functions of their derivatives, although $div\mathbf{E} = div\mathbf{B} = 0$.

This allows in principle to consider the two Frobenius integrability conditions

$$\mathbf{d}\eta \wedge \eta = (\mathbf{E}.curl\,\mathbf{E})\,\omega = 0, \quad \mathbf{d}\beta \wedge \beta = (\mathbf{B}.curl\,\mathbf{B})\,\omega = 0$$

as compatible with the nonlinear static equations, so, static electric and magnetic *helicities*, which are not allowed by Maxwell static equations, not to be excluded from the very beginning. For example, the vector fields X, satisfuing $X \times curl\,X = 0, div\,X = 0$, known as Beltrami vector fields, exist and are of definite interest in fluid mechanics and optics [22].

Following this line of consideration we find

$$L_{\mathbf{E}\times\mathbf{B}}\,\omega = \mathbf{d}i_{\mathbf{E}\times\mathbf{B}}\,\omega + i_{\mathbf{E}\times\mathbf{B}}\mathbf{d}\,\omega = \mathbf{d}i_{\mathbf{E}\times\mathbf{B}}\,\omega = div(\mathbf{E}\times\mathbf{B})\,\omega = (\mathbf{B}.curl\,\mathbf{E} - \mathbf{E}.curl\,\mathbf{B})\,\omega.$$

So, Poynting theorem suggests to write down (denoting $\xi = ct$)

$$L_{\mathbf{E}\times\mathbf{B}}\,\omega = -\frac{\partial}{\partial\xi}\frac{\mathbf{E}^2 + \mathbf{B}^2}{2}\,\omega,$$

i.e.,

$$\mathbf{B}.curl\,\mathbf{E} - \mathbf{E}.curl\,\mathbf{B} = -\frac{\partial}{\partial\xi}\frac{\mathbf{E}^2 + \mathbf{B}^2}{2}, \quad \rightarrow \quad \mathbf{B}.\left(curl\,\mathbf{E} + \frac{\partial\mathbf{B}}{\partial\xi}\right) = \mathbf{E}.\left(curl\,\mathbf{B} - \frac{\partial\mathbf{E}}{\partial\xi}\right).$$

These relations say: the difference of the two mutual *local cross-helicities* $\mathbf{B}.curl\,\mathbf{E}$ and $\mathbf{E}.curl\,\mathbf{B}$ deforms the volume form $\omega = dx \wedge dy \wedge dz$ by the ξ-derivative of the energy-density. Moreover,

two of the Maxwell equations are sufficient for this, and if the two fields are nonzero only inside compact 3d-region $\mathbb{A} \subset \mathbb{R}^3$, for each t, then

$$L_{\mathbf{E}\times\mathbf{B}}\,\omega = div(\mathbf{E}\times\mathbf{B})\omega = (\mathbf{B}.curl\,\mathbf{E} - \mathbf{E}.curl\,\mathbf{B})\omega = \mathbf{d}(\beta\wedge\eta).$$

Now the Stokes theorem with respect to $\mathbb{A} \subset \mathbb{R}^3$ leads to zero of the integral $\int_{\mathbf{R}^3}(L_{\mathbf{E}\times\mathbf{B}})\,\omega$, so,

$$\frac{\partial}{\partial\xi}\int_{\mathbb{R}^3}\frac{\mathbf{E}^2+\mathbf{B}^2}{2}\,\omega = 0,$$

i.e., the integral energy is conserved.

The relations considered suggest some connection with the concepts of *absolute* and *relative* integral invariants of a vector field X on a manifold M introduced and used by E.Cartan [23]: these are differential forms $\alpha \in \Lambda(M)$ satisfying respectively the relations $i(X)\alpha = 0, i(X)\mathbf{d}\alpha = 0$, leading to $L_X\alpha = 0$, and just $i(X)\mathbf{d}\alpha = 0$. Our relations may be considered as corresponding extensions: *a vector field* $\rightarrow$ *vector valued multivector field* and *a differential form* $\rightarrow$ *vector valued differential form* making use of the mentioned in Sec.4.1 extension of the Lie derivative of a differential form along multivector fields. The new moment in our extension is that we consider vector valued multivectors along which vector valued forms to be differentiated with respect to some bilinear map $\varphi : V\times V \rightarrow W$, where W is appropriately determined vector space.

In general, we note that, the thriple $(V, W; \varphi)$ determines possible interactions among the subsystems of the field object considered, which subsystems are formally represented by the vector components of the multivector (in our case $\bar{\mathbf{\Omega}}$) and the vector components of the (multi)differential form (in our case $\mathbf{\Omega}$).

5 Conclusion

Getting knowledge of the internal compatibility and external stability of a physical object is being done by measuring the corresponding to these physical appearances appropriate physical quantities. Such physical quantities may vary in admissible, or not admissible degree: in the first case we talk about admissible changes, and in the second case we talk about changes leading to destruction of the object. Formally, this is ususlly checked by calculating the flow of the formal image of the (sub)system considered through its appropriately modeled change, as it is seen, e.g., in (8),(9),(11), i.e., by means of finding corresponding *differential self flows of the subsystems*, e.g., $i_{\bar{F}}\mathbf{d}F$, and *differential mutual flows among the subsystems*, e.g., $i_{\bar{F}}\mathbf{d}*F$. Since every measuring process requires stress-energy-momentum transfering between the object studied and the measuring system, the role of finding corresponding *tensor* representatives of these change-objects and the corresponding flows is of serious importance. Therefore, having adequate stress-energy-momentum for the considered case, the clearly individualized tensor members of its divergence represent qualitatively and quantitatively important aspects of the *intrinsic interacting dynamical nature* of the object considered. This view motivated the above given approach to find appropriate description of electromagnetic field objects.

The existing knowledge about the structure and internal dynamics of free electromagnetic field objects made us assume the notion for *two partner-fields internal structure*, formally represented by $(F, *F)$ on Minkowski space-time. Each of these two partner-fields is built of the two *formal constituents* $(\mathbf{E}, \mathbf{B})$, and each partner-field is able to carry local stress-energy-momentum, allowing local "intercomunication" between its two constituents during the local interaction with its partner-field. The two subsystems carry equal local energy-momentum densities, and realize local mutual energy exchange *without available interaction energy.* Moreover, they strictly respect each other:

the exchange is *simultaneous* and in *equal quantities*, so, each of the two partner-fields keeps its identity and recognizability. The corresponding internal dynamical structure appropriately unifies translation and rotation through unique space-time propagations as a whole with the fundamental velocity. All Maxwell solutions are duly respected. The new nonlinear solutions, i.e., those satisfying $\mathbf{d}F \neq 0, \mathbf{d} * F \neq 0$, are *time-stable, they admit FINITE SPATIAL SUPPORT, and minimize the relation* $I_1^2 + I_2^2 \geq 0$. It deserves noting here that the obtained relation $I_1^2 + I_2^2 = 0$ for the nonlinear solutions is equivalent to $I_1 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \mathbf{B}^2 - \mathbf{E}^2 = 0, \quad I_2 = \frac{1}{2}F_{\mu\nu}(*F)^{\mu\nu} = 2\mathbf{E}.\mathbf{B} = 0$ (for details see [21]).

The admitted solutions with spatially finite support are of *photon-like nature*:

-they are time-stable,

-they demonstrate intrinsically compatible translational-rotational dynamical structure,

-they propagate translationally as a whole with the velocity of light,

-they carry finite energy-momentum and intrinsically determined integral characteristic $\mathfrak{h}$ of action nature through naturally available appropriate scale factor $\mathcal{L}_o = const$ [21,pp.233] carrying physical dimension of length,

-their integral energy E satisfies relation of the form identical to the Planck formula $E.T = \mathfrak{h}$ [21,pp.230-231].

Some of these nonlinear solutions of (12) look like:

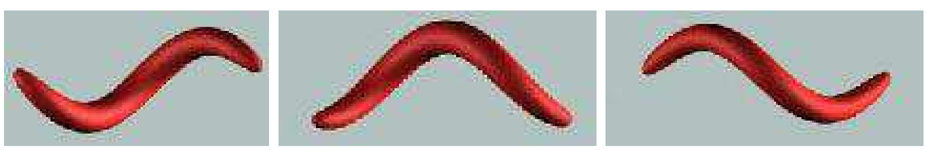

Figure 1: Theoretical example with clock-wise rotation, translation: left to right

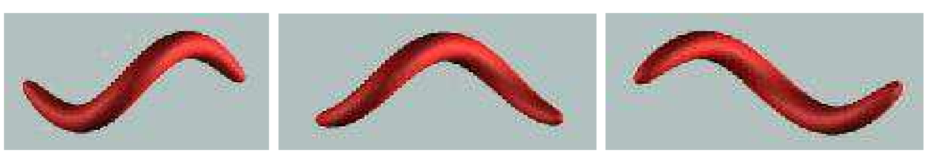

Figure 2: Theoretical example with anticlock-wise rotation, translation: left to right

As we mentioned above, the straight line size along translational propagation of each of these finite helical-like objects is $2\pi\mathcal{L}_o = const$, $T = \frac{2\pi\mathcal{L}_o}{c}$, so, $\mathfrak{h}$ is an integral Lorentz invariant action characteristic of any solution of this class, meaning: there is specific propagational action demonstrated during the intrinsically defined time period $\frac{2\pi\mathcal{L}_o}{c}$, where c is the invariant speed of translational propagation as a whole.

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